

Generalized Darboux transformation and higher-order rogue wave solutions to the Manakov system.

Serge P. Mukam^{1,2,*}, Victor K. Kuetché^{1,2,3,4} and Thomas B. Bouetou^{1,2,3,4*}

¹*Ecole Nationale Supérieure Polytechnique, University of Yaounde I, P.O. Box. 8390, Cameroon*

²*Department of Physics, Faculty of Science, University of Yaounde I, P.O. Box. 812, Cameroon*

³*Centre d'Excellence en Technologies de l'Information et de la Communication (CETIC),
University of Yaounde I, P.O. Box 812, Yaounde, Cameroon and*

⁴*The Abdus Salam International Centre for Theoretical Physics (ICTP), Strada Costiera, Trieste 11-I-34151, Italy
(Dated: August 10, 2010)*

In this paper, we construct a generalized recursive Darboux transformation of a focusing vector nonlinear Schrödinger equation known as the Manakov system. We apply this generalized recursive Darboux transformation to the Lax-pairs of this system in view of generating the Nth-order vector generalization rogue wave solutions with the same spectral parameter through a direct iteration rule. As a result, we discuss the first, second and third-order vector generalization rogue wave solutions while illustrating these features with some depictions. We show that higher-order rogue wave solutions depend on the values of their free parameters.

* Corresponding author: sergemukam@yahoo.fr

I. INTRODUCTION

Understanding the behavior and the dynamics of natural phenomena stand to be worth fundamental. That is why it has been one of the most challenging aspects of modern science and technology in studying nonlinear nature of system. The importance of nonlinearity has been well-appreciated for many years, because nonlinearity is a fascinating occurrence of nature in the context of large amplitude waves or high-intensity laser pulses observed in various fields. This fascinating subject has branched out in almost all areas of science, and its applications are percolating through the whole science. In general, nonlinear evolution equations exhibiting a wide range of high complexities in terms of different linear and nonlinear effects model nonlinear phenomena. Nonlinear science has experienced an explosive growth by the invention of several exciting and fascinating new concepts in the past few decades, such as solitons, dispersion-managed solitons, dromions, rogue waves, among others [1–4].

Rogue waves, also called freak waves, giant waves or killer waves have attracted considerable attentions. A rogue wave is a large-amplitude local wave, short-lived wave, meet in an ocean that appears from nowhere and disappears without a trace. Rogue waves appear not only in oceanic conditions [5], but also in optics [6], superfluids [7], Bose-Einstein condensates [8] and in the form of capillary waves [9]. In any of these disciplines, new studies of rogue waves enrich the concept and lead to progress toward a comprehensive understanding of this still mysterious phenomenon. The first-order rational solution for the nonlinear Schrödinger equation (NLSE) was given by Peregrine [10]. Akhmediev and co-workers have calculated the simplest rogue wave solutions for the NLSE [11, 12]. The construction of higher-order analogues is actually a challenging problem. The construction of higher-order rogue wave solutions needs a simple approach which is the generalized Darboux transformation (DT). This approach was proposed by Guo et al [13].

The DT, originating from the work of Darboux [14] on the Sturm-Liouville equation, is a powerful method for constructing solutions for integrable systems such as the NLSE, the Korteweg-de Vries (KdV) equation, the Kadomtsev-Petviashvili equation, the Davey-Stewartson equation and the Toda lattice equation just to name a few. But the original DT is not applicable directly to obtain the rogue wave solutions for the nonlinear wave equations [13]. Matveev [15] introduced the so-called generalized DT and the positon solutions were calculated for the celebrated KdV equation. Recently, Guo et al. [13] re-examined Matveev's generalized DT and proposed a new approach to derive the generalized DT for the KdV and the NLS equations.

In this work, we discuss the Guo et al [13]'s approach to a focusing vector NLSE (VNLSE), known as Manakov system given as follows

$$iu_t + \frac{1}{2}u_{xx} + u(|u|^2 + |v|^2) = 0, \quad (1)$$

$$iv_t + \frac{1}{2}v_{xx} + v(|u|^2 + |v|^2) = 0, \quad (2)$$

with x and t being two independent variables, $u(x, t)$ and $v(x, t)$ standing for complex envelop of two field components. This system has many physical significant applications such as the propagation in elliptically birefringent optical fibers [16] and for modeling crossing sea waves [17].

We consider in this paper the VNLS equations in the anomalous dispersion regime. The aim of this work is to construct Nth-order rogue wave solutions of the previous system. The main tool is the generalized DT. Based on the Darboux matrix method [18, 19], we iterate the generalized DT of equations (1) and (2) and work out a formula for generation of higher-order rogue wave solutions. It is worth noting that the rogue wave solutions of this system have early been constructed in ref. [20], using however a non-recursive Darboux transformation up to second-order. Thus, the organization of this paper is settled as follows. In section 2, we first propose, by the Darboux matrix method and the Lax-pairs, a DT for the focusing VNLSE, we then use this DT to derive, via the Taylor expansion of the solution to the Lax-pairs, the generalized DT for the focusing VNLSE. Also, we provide, through iterations of the DT, formulae for Nth-order rogue wave solutions of the focusing VNLSE. In section 3, we consider the dynamics for these solutions from the first to third-order rogue waves and show their interesting structures. The last section is devoted to a brief conclusion.

II. GENERALIZED DARBOUX TRANSFORMATION

Equations (1) and (2) can be cast into a 3×3 linear eigenvalue problem due to integrability [20]

$$\mathbf{R}_x = \mathbf{U}\mathbf{R}, \quad \mathbf{R}_t = \mathbf{V}\mathbf{R}, \quad (3)$$

where,

$$\mathbf{U} = \lambda \mathbf{E} + \mathbf{Q}, \quad \mathbf{V} = \frac{3}{2}\lambda^2 \mathbf{E} + \frac{3}{2}\lambda \mathbf{Q} + \frac{i}{2}\sigma_3(\mathbf{Q}_x - \mathbf{Q}^2), \quad (4)$$

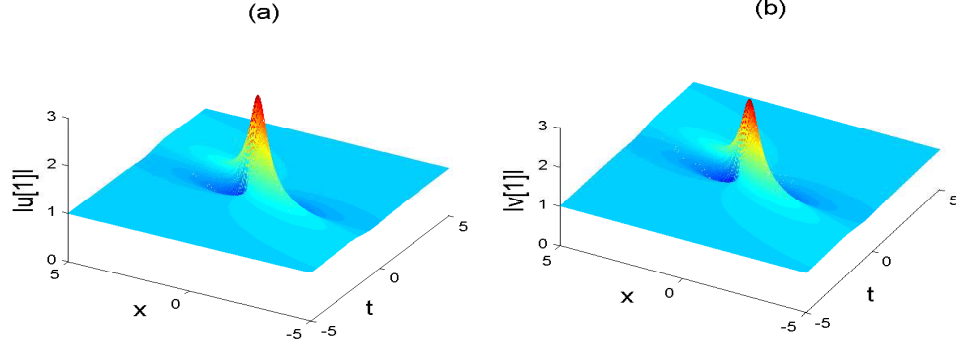


FIG. 1. First-order rogue waves $u[1]$ and $v[1]$ depicted with the following parameters (a) $a = 1$, $b = 1/4$, and (b) $a = 1/4$, $b = 1$, $\alpha = 1$.

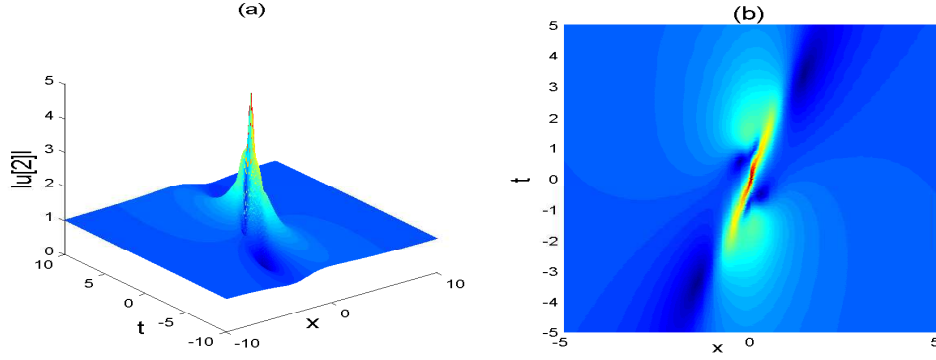


FIG. 2. Second-order composite rogue wave $u[2]$ depicted with the following parameters: $a = 1$, $b = 0$, $c_1 = d_1 = 0$ and $\alpha = 1$. Panel (a) represents the 3-D perspective and the panel (b) stands for the density plot of the 3-D representation.

with, $\mathbf{E} = \text{diag}(-2i, i, i)$, $\sigma_3 = \text{diag}(1, -1, -1)$, $\mathbf{Q} = \begin{pmatrix} 0 & -u & -v \\ u^* & 0 & 0 \\ v^* & 0 & 0 \end{pmatrix}$.

Here $\mathbf{R} = (r(x, t), s(x, t), w(x, t))^T$ (T means a matrix transpose), u and v are potentials, λ is the spectral parameter, u^* and v^* denote complex conjugate of u and v . Through direct calculation, one can directly obtain (1) and (2) by using the zero curvature equation $\mathbf{U}_t - \mathbf{V}_x + \mathbf{UV} - \mathbf{VU} = \mathbf{0}$.

Let $\mathbf{R}_1 = (r_1, s_1, w_1)^T$ be a solution of the Lax-pairs given in equation (3) with $u = u[0]$, $v = v[0]$ and $\lambda = \lambda_1$. The classical DT of the Ablowitz-Kaup-Newell-Segur (AKNS) spectral problem [18] allows us to write the following formulas:

$$\mathbf{R}[1] = T[1]\mathbf{R}, \quad T[1] = \lambda_1 \mathbf{I} - H[0]\Lambda_1 H[0]^{-1}, \quad (5)$$

$$u[1] = u[0] + 2i(\lambda - \lambda^*) \frac{r_1[0]s_1[0]^*}{|r_1[0]|^2 + |s_1[0]|^2 + |w_1[0]|^2}, \quad (6)$$

$$v[1] = v[0] + 2i(\lambda - \lambda^*) \frac{r_1[0]w_1[0]^*}{|r_1[0]|^2 + |s_1[0]|^2 + |w_1[0]|^2}, \quad (7)$$

which satisfy

$$\mathbf{R}[1]_x = \mathbf{U}[1]\mathbf{R}[1], \quad \mathbf{R}[1]_t = \mathbf{V}[1]\mathbf{R}[1], \quad (8)$$

where $\mathbf{R}_1[0] = T[0]\mathbf{R}_1$, $r_1[0] = r_1$, $s_1[0] = s_1$, $w_1[0] = w_1$, $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $H[0] = \begin{pmatrix} r_1[0] & s_1[0]^* & w_1[0]^* \\ s_1[0] & -r_1[0]^* & 0 \\ w_1[0] & 0 & -r_1[0]^* \end{pmatrix}$,

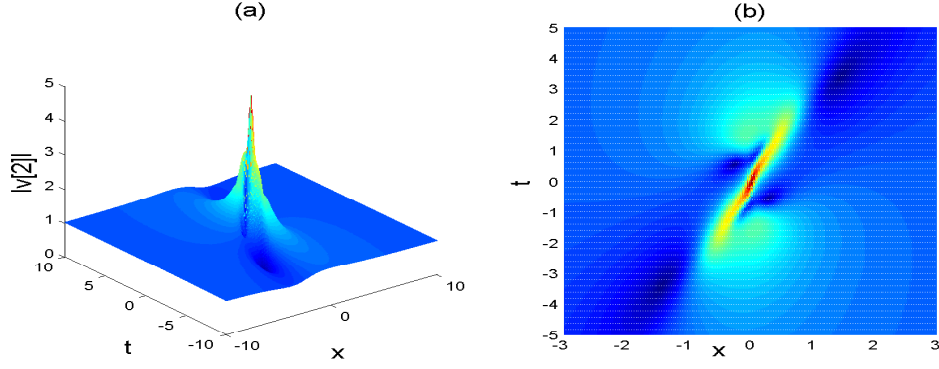


FIG. 3. Second-order composite rogue wave $v[2]$ depicted with the following parameters: $a = 0$, $b = 1$, $c_1 = d_1 = 0$ and $\alpha = 1$. Panel (a) represents the 3-D perspective and the panel (b) stands for the density plot of the 3-D representation.

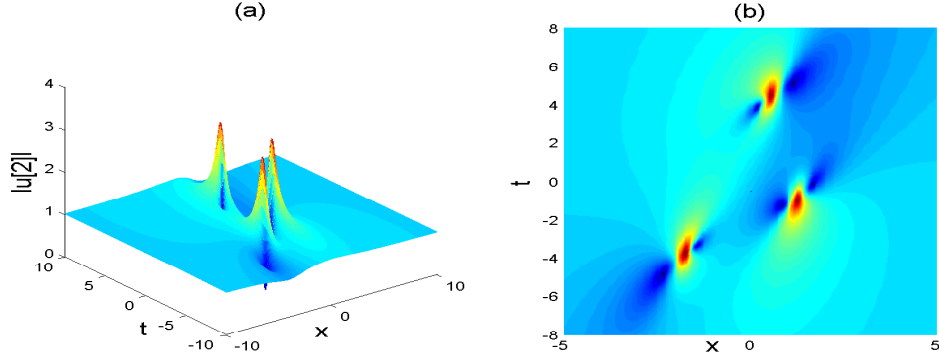


FIG. 4. Second-order rogue wave $u[2]$ depicted with the following parameters: $a = 1$, $b = 0$, $c_1 = d_1 = 25$ and $\alpha = 1$. Panel (a) represents the 3-D perspective and the panel (b) stands for the density plot of the 3-D representation.

and $\Lambda_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1^* & 0 \\ 0 & 0 & \lambda_1^* \end{pmatrix}$. The quantities $\mathbf{U}[1]$ and $\mathbf{V}[1]$ have the same form as \mathbf{U} and \mathbf{V} except that the old potentials u and v are replaced by the new ones $u[1]$ and $v[1]$. The quantity T stands for the Darboux matrix.

If N distinct basic solutions $\mathbf{R}_k = (r_k, s_k, w_k)^T$ ($k = 1, 2, 3, \dots, N$) of the Lax-pairs expressed by equation (3) at $\lambda = \lambda_k$ ($k = 1, 2, 3, \dots, N$) are given, the DT can be repeated N times. Then, the N th-step DT for the VNLSE (1) and (2) is

$$\mathbf{R}[N] = T[N]T[N-1]\dots T[1]\mathbf{R}, \quad T[k] = \lambda \mathbf{I} - H[k-1]\Lambda_k H[k-1]^{-1}, \quad (9)$$

$$u[N] = u[N-1] + 2i(\lambda - \lambda^*) \frac{r_N[N-1]s_N[N-1]^*}{|r_N[N-1]|^2 + |s_N[N-1]|^2 + |w_N[N-1]|^2}, \quad (10)$$

$$v[N] = v[N-1] + 2i(\lambda - \lambda^*) \frac{r_N[N-1]w_N[N-1]^*}{|r_N[N-1]|^2 + |s_N[N-1]|^2 + |w_N[N-1]|^2}, \quad (11)$$

where

$$H[k-1] = \begin{pmatrix} r_k[k-1] & s_k[k-1]^* & w_k[k-1]^* \\ s_k[k-1] & -r_k[k-1]^* & 0 \\ w_k[k-1] & 0 & -r_k[k-1]^* \end{pmatrix}, \quad \Lambda_k = \begin{pmatrix} \lambda_k & 0 & 0 \\ 0 & \lambda_k^* & 0 \\ 0 & 0 & \lambda_k^* \end{pmatrix},$$

$\mathbf{R}_k[k-1] = (r_k[k-1], s_k[k-1], w_k[k-1])^T = \mathbf{R}_k[k-1]$ and $\mathbf{R}[k-1] = T[k-1]T[k-2]\dots T[1]\mathbf{R}$.

According to the above classical DT, we derive the generalized DT for the system of equations (1) and (2). We start with the assumption that

$$R_1 = R_1(\lambda_1 + \epsilon), \quad (12)$$

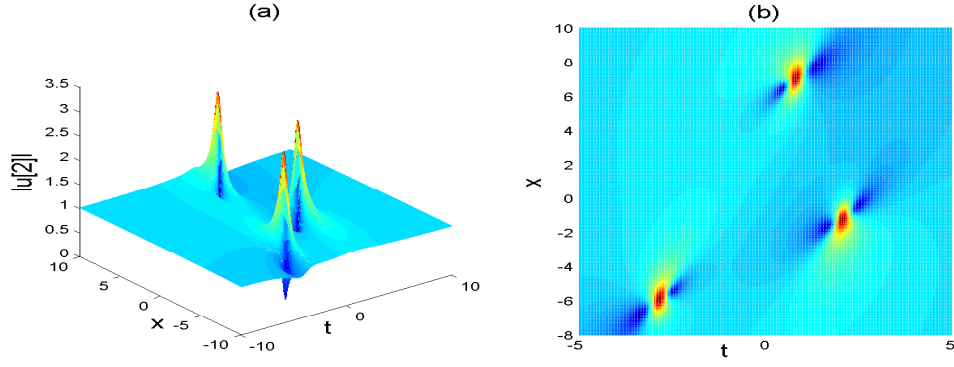


FIG. 5. Second-order rogue wave $u[2]$ depicted with the following parameters: $a = 1$, $b = 0$, $c_1 = d_1 = 100$ and $\alpha = 1$. Panel (a) represents the 3-D perspective and the panel (b) stands for the density plot of the 3-D representation.

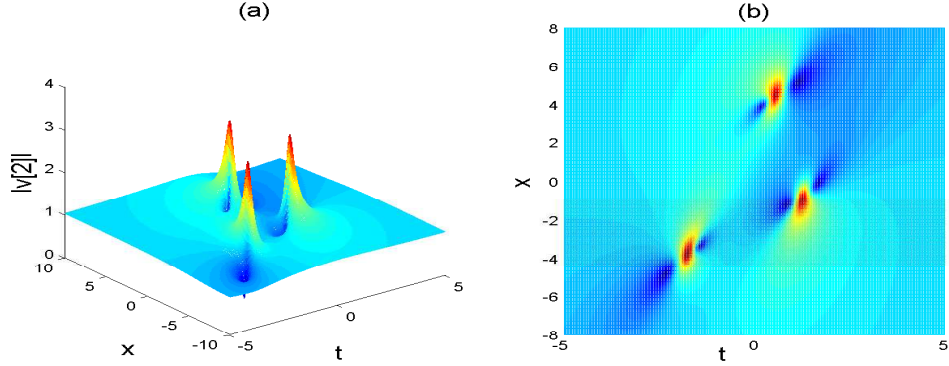


FIG. 6. Second-order rogue wave $v[2]$ depicted with the following parameters: $a = 0$, $b = 1$, $c_1 = d_1 = 25$ and $\alpha = 1$. Panel (a) represents the 3-D perspective and the panel (b) stands for the density plot of the 3-D representation.

is a particular solution of the Lax-pairs of equation (3). The constant ϵ being a small parameter. Expanding R_1 in a Taylor series gives

$$R_1 = R_1^{[0]} + R_1^{[1]}\epsilon + R_1^{[2]}\epsilon^2 + \dots + R_1^{[N]}\epsilon^N + \dots, \quad (13)$$

where $R_1^{[k]} = \frac{1}{k!} \frac{\partial^k}{\partial \epsilon^k} R_1|_{\epsilon=0}$ ($k = 1, 2, 3, \dots$).

1. The first-step of the method.

From the above assumption, it is easy to find that $R_1^{[0]}$ is a solution for the Lax-pairs of equation (3) with $u = u[0]$ and $v = v[0]$ at $\lambda = \lambda_1$. The first-step DT for the system (1) and (2) is expressed as

$$\mathbf{R}[1] = T[1]\mathbf{R}, \quad T[1] = \lambda_1 \mathbf{I} - H[0]\Lambda_1 H[0]^{-1}, \quad (14)$$

$$u[1] = u[0] + 2i(\lambda - \lambda^*) \frac{r_1[0]s_1[0]^*}{|r_1[0]|^2 + |s_1[0]|^2 + |w_1[0]|^2}, \quad (15)$$

$$v[1] = v[0] + 2i(\lambda - \lambda^*) \frac{r_1[0]w_1[0]^*}{|r_1[0]|^2 + |s_1[0]|^2 + |w_1[0]|^2}, \quad (16)$$

where

$$\mathbf{R}_1[0] = T[0]\mathbf{R}_1, \quad r_1[0] = r_1, \quad s_1[0] = r_1, \quad r_1[0] = r_1, \quad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H[0] = \begin{pmatrix} r_1[0] & s_1[0]^* & w_1[0]^* \\ s_1[0] & -r_1[0]^* & 0 \\ w_1[0] & 0 & -r_1[0]^* \end{pmatrix},$$

$$\text{and } \Lambda_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1^* & 0 \\ 0 & 0 & \lambda_1^* \end{pmatrix}.$$

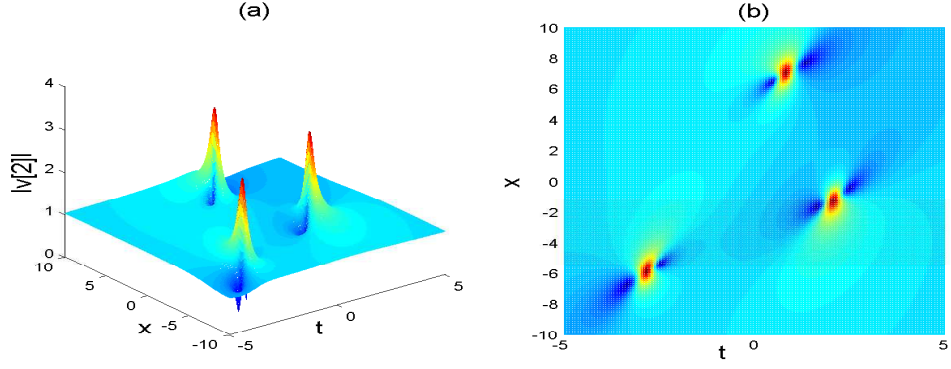


FIG. 7. Second-order rogue wave $v[2]$ depicted with the following parameters: $a = 0$, $b = 1$, $c_1 = d_1 = 100$ and $\alpha = 1$. Panel (a) represents the 3-D perspective and the panel (b) stands for the density plot of the 3-D representation.

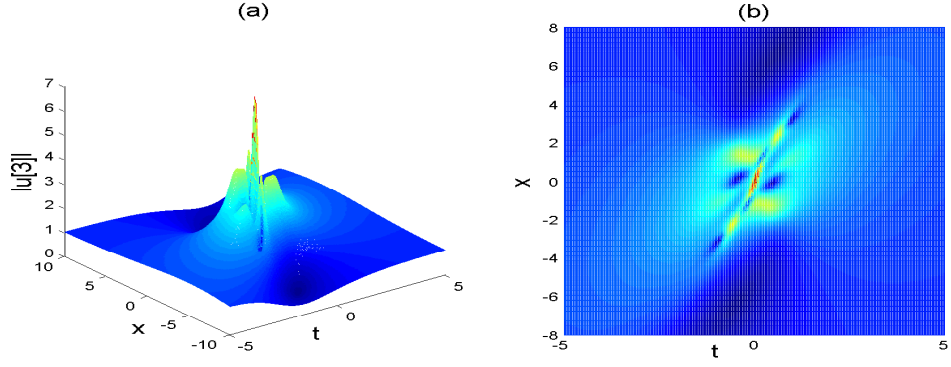


FIG. 8. Third-order composite rogue wave $u[3]$ depicted with the following parameters: $a = 1$, $b = 0$, $c_1 = d_1 = c_2 = d_2 = 0$ and $\alpha = 1$. Panel (a) represents the 3-D perspective and the panel (b) stands for the density plot of the 3-D representation.

2. The second-step of the method.

With

$$\lim_{\epsilon \rightarrow 0} \frac{T[1]_{|\lambda=\lambda_1+\epsilon R_1}}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{(\epsilon + T_1[1])R_1}{\epsilon} = R_1^{[0]} + T_1[1]R_1^{[1]} \equiv R_1[1], \quad (17)$$

we find a solution to the Lax-pairs of equation (3) with $u[2]$ and $v[2]$ and $\lambda = \lambda_1$. This allows us to go to the second step DT, namely,

$$R[2] = T[2]T[1]R, \quad T[2] = \lambda_1 \mathbf{I} - H[1]\Lambda_2 H[1]^{-1}, \quad (18)$$

$$u[2] = u[1] + 2i(\lambda - \lambda^*) \frac{r_1[1]s_1[1]^*}{|r_1[1]|^2 + |s_1[1]|^2 + |w_1[1]|^2}, \quad (19)$$

$$v[2] = v[1] + 2i(\lambda - \lambda^*) \frac{r_1[1]w_1[1]^*}{|r_1[1]|^2 + |s_1[1]|^2 + |w_1[1]|^2}, \quad (20)$$

where $(r_1[1], s_1[1], w_1[1])^T = R_1[1]$, $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $H[1] = \begin{pmatrix} r_1[1] & s_1[1]^* & w_1[1]^* \\ s_1[1] & -r_1[1]^* & 0 \\ w_1[1] & 0 & -r_1[1]^* \end{pmatrix}$,

and $\Lambda_2 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1^* & 0 \\ 0 & 0 & \lambda_1^* \end{pmatrix}$.

3. The third-step of the method.

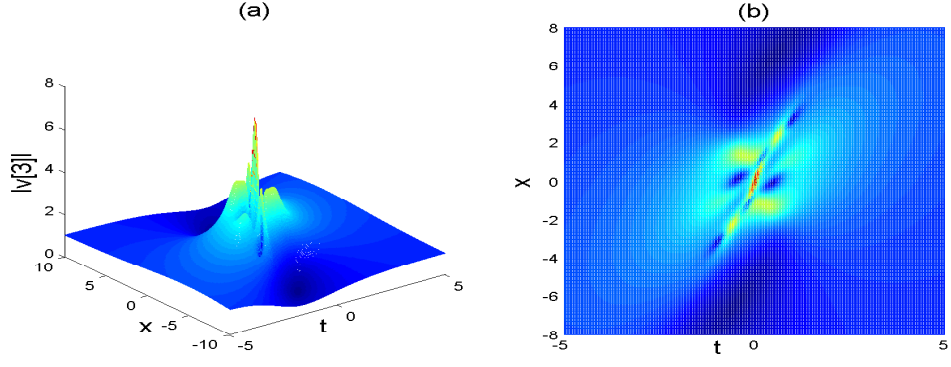


FIG. 9. Third-order composite rogue wave $v[3]$ depicted with the following parameters: $a = 0$, $b = 1$, $c_1 = d_1 = c_2 = d_2 = 0$ and $\alpha = 1$. Panel (a) represents the 3-D perspective and the panel (b) stands for the density plot of the 3-D representation.

Similarly, the following limit

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{[T[2]T[1]]|_{\lambda=\lambda_1+\epsilon} R_1}{\epsilon^2} &= \lim_{\epsilon \rightarrow 0} \frac{(\epsilon + T_1[2])(\epsilon + T_1[1])R_1}{\epsilon^2} \\ &= R_1^{[0]} + (T_1[2] + T_1[1])R_1^{[1]} + T_1[2]T_1[1]R_1^{[2]} \equiv R_1[2], \end{aligned} \quad (21)$$

provides us with a non-trivial solution for the Lax-pairs of equation (3) with $u[3]$, $v[3]$ and $\lambda = \lambda_1$. Then, the third-step generalized DT can be given as follows

$$R[3] = T[3]T[2]T[1]R, \quad T[3] = \lambda_1 \mathbf{I} - H[2]\Lambda_3 H[2]^{-1}, \quad (22)$$

$$u[3] = u[2] + 2i(\lambda - \lambda^*) \frac{r_1[2]s_1[2]^*}{|r_1[2]|^2 + |s_1[2]|^2 + |w_1[2]|^2}, \quad (23)$$

$$v[3] = v[2] + 2i(\lambda - \lambda^*) \frac{r_1[2]w_1[2]^*}{|r_1[2]|^2 + |s_1[2]|^2 + |w_1[2]|^2}, \quad (24)$$

where $(r_1[2], s_1[2], w_1[2])^T = R_1[2]$, $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $H[2] = \begin{pmatrix} r_1[2] & s_1[2]^* & w_1[2]^* \\ s_1[2] & -r_1[2]^* & 0 \\ w_1[2] & 0 & -r_1[2]^* \end{pmatrix}$,

and $\Lambda_3 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1^* & 0 \\ 0 & 0 & \lambda_1^* \end{pmatrix}$.

4. The N th-step generalized DT.

Continuing the above process and combining all the Darboux matrices, an N th-step generalized DT is considered as follows:

$$R_1[N-1] = R_1^{[0]} + \sum_{k=1}^{N-1} T_1[k]R_1^{[1]} + \sum_{k=1}^{N-1} \sum_{l=1}^{k-1} T_1[k]T_1[l]R_1^{[2]} + \dots + T_1[N-1]T_1[N-2]\dots T_1[1]R_1^{[N-1]}, \quad (25)$$

$$R[N] = T[N]T[N-1]\dots T[1]R, \quad T[k] = \lambda_1 \mathbf{I} - H[k-1]\Lambda_k H[k-1]^{-1}, \quad (26)$$

$$u[N] = u[N-1] + 2i(\lambda - \lambda^*) \frac{r_1[N-1]s_1[N-1]^*}{|r_1[N-1]|^2 + |s_1[N-1]|^2 + |w_1[N-1]|^2}, \quad (27)$$

$$v[N] = v[N-1] + 2i(\lambda - \lambda^*) \frac{r_1[N-1]w_1[N-1]^*}{|r_1[N-1]|^2 + |s_1[N-1]|^2 + |w_1[N-1]|^2}, \quad (28)$$

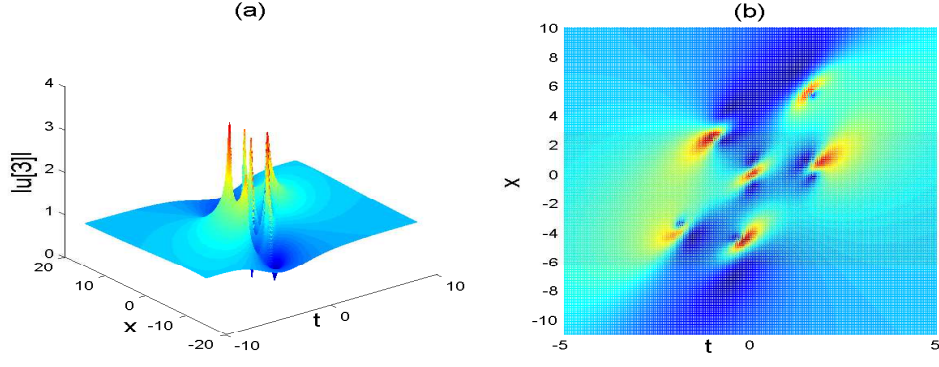


FIG. 10. Third-order rogue wave $u[3]$ depicted with the following parameters: $a = 1$, $b = 0$, $c_1 = d_1 = 0$, $c_2 = d_2 = 100$ and $\alpha = 1$. Panel (a) represents the 3-D perspective and the panel (b) stands for the density plot of the 3-D representation

where $(r_1[N-1], s_1[N-1], w_1[N-1])^T = R_1[N-1] H[k-1] = \begin{pmatrix} r_1[k-1] & s_1[k-1]^* & w_1[k-1]^* \\ s_1[k-1] & -r_1[k-1]^* & 0 \\ w_1[k-1] & 0 & -r_1[k-1]^* \end{pmatrix}$,

$$\Lambda_k = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1^* & 0 \\ 0 & 0 & \lambda_1^* \end{pmatrix}, \text{ and } \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The formulae given by equations (26)-(28) are a recursive formula of the Nth-order generalized DT for the Manakov system (1) and (2). Although it is possible to give the $(3N) \times (3N)$ determinant representation by using the so-called crum theorem [21]. But we prefer to use a recursive formula, because it is easy to construct higher-order rogue wave solutions with the computer. Some interesting higher-order rogue wave solutions are obtained for the Manakov system in the following section.

III. ROGUE WAVE SOLUTIONS

In order to obtain the rogue wave solution, we start with the following seed solutions of the system (1) and (2) as

$$u[0] = ae^{i\beta t}, \quad v[0] = be^{i\beta t}, \quad (29)$$

with $\beta = a^2 + b^2$, a and b are real constants. Then the basic solution for the Lax-pairs of equation (3) with $u[0]$, $v[0]$ and λ holds

$$R_1 = \begin{pmatrix} (m_1 e^{\eta_1 + \eta_2} - m_2 e^{\eta_1 - \eta_2}) e^{\frac{i\beta t}{2}} \\ \tau_1 (m_2 e^{\eta_1 + \eta_2} - m_1 e^{\eta_1 - \eta_2}) e^{-\frac{i\beta t}{2}} \\ \tau_2 (m_2 e^{\eta_1 + \eta_2} - m_1 e^{\eta_1 - \eta_2}) e^{-\frac{i\beta t}{2}} \end{pmatrix}, \quad (30)$$

where $m_1 = \left(\frac{\lambda - \sqrt{\beta + \lambda^2}}{\lambda^2 + \beta} \right)^{\frac{1}{2}}$, $m_2 = \left(\frac{\lambda + \sqrt{\beta + \lambda^2}}{\lambda^2 + \beta} \right)^{\frac{1}{2}}$, $\eta_1 = 2i\alpha(\frac{1}{\alpha}x + \lambda(\lambda + 2)t)$, $\tau_1 = \frac{a}{\sqrt{\beta}}$, $\tau_2 = \frac{b}{\sqrt{\beta}}$,

$$\eta_2 = i\alpha\sqrt{\beta + \lambda^2} \left(\frac{1}{\alpha}x + (\lambda(\lambda + 2) - \frac{\beta}{2})t + \sum_{j=1}^N (c_j + id_j)\varepsilon^{2j} \right).$$

Here the constant ε is a small parameter.

Let $\lambda = i\sqrt{\beta}(1 + \varepsilon^2)$, expanding the vector function $R_1(\varepsilon)$ at $\varepsilon = 0$, we obtain

$$R_1(\varepsilon) = R_1^{[0]} + R_1^{[1]}\varepsilon^2 + R_1^{[2]}\varepsilon^4 + \dots, \quad (31)$$

where,

$$R_1^{[0]} = \begin{pmatrix} r_1^0 \\ s_1^0 \\ w_1^0 \end{pmatrix}, R_1^{[1]} = \begin{pmatrix} r_1^1 \\ s_1^1 \\ w_1^1 \end{pmatrix}, R_1^{[2]} = \begin{pmatrix} r_1^2 \\ s_1^2 \\ w_1^2 \end{pmatrix} \dots, \text{ and } (r_1^{[i-1]}, s_1^{[i-1]}, w_1^{[i-1]}) \text{ (i=1,2,3) are given in appendix.}$$

It is clear that $R_1^{[0]}$ is a solution of the Lax pairs (3) at $u[0] = ae^{i\beta t}$, $v[0] = be^{i\beta t}$ and $\lambda = i\sqrt{\beta}(1 + \varepsilon^2)$. Hence from the formulae (15) and (16), we arrive at

$$u[1] = ae^{i\beta t} \left(1 + \frac{F_1 + iH_1}{D_1} \right), \quad v[1] = be^{i\beta t} \left(1 + \frac{F_1 + iH_1}{D_1} \right), \quad (32)$$

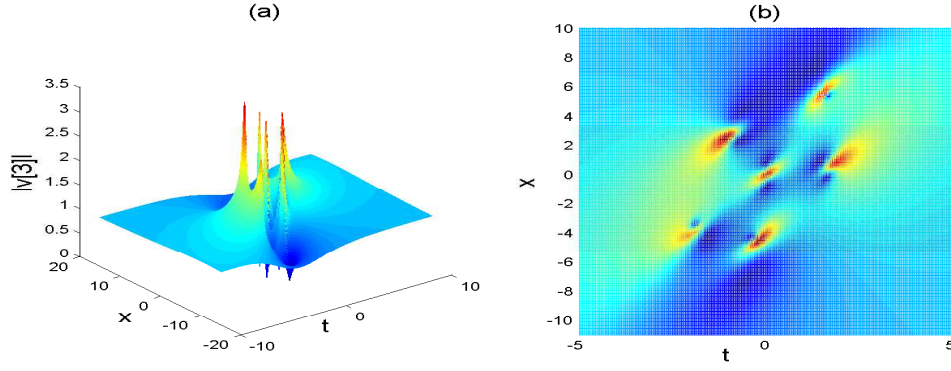


FIG. 11. Third-order rogue wave $v[3]$ depicted with the following parameters: $a = 0$, $b = 1$, $c_1 = d_1 = 0$, $c_2 = d_2 = 100$ and $\alpha = 1$. Panel (a) represents the 3-D perspective and the panel (b) stands for the density plot of the 3-D representation

where

$$F_1 = -4\sqrt{\beta}\alpha(4\beta x^2 - 12\alpha\beta^2 tx + \alpha^2\beta^2(9\beta + 16)t^2 - 1),$$

$$H_1 = 32\alpha\sqrt{\beta}t, D_1 = \frac{2}{\beta}(4\beta x^2 - 12\alpha\beta^2 tx + \alpha^2\beta^2(9\beta + 16)t^2 + 1).$$

The solutions $u[1]$ and $v[1]$ here stand for the vector generalized first-order rogue wave solutions for the Manakov system (1) and (2). It is important to remark that, $u[1]$ and $v[1]$ are merely proportional. These solutions are depicted in figure 1. It is also important to note that the solutions obtained here for the Manakov system resemble those obtained in ref [20].

Then, using the matrices $R_1^{[0]}$ and $R_1^{[1]}$ given in appendix, and substituting them into the expression of equation (17), we obtain the matrix $R_1[1]$ which elements are also given in appendix. The matrix elements $r_1[1]$, $s_1[1]$ and $w_1[1]$ are then substituted into equations (19) and (20), which give rise to the second-order vector generalization rogue wave solutions of the Manakov system (1) and (2).

The second-order rogue wave solution possesses two free parameters c_1 and d_1 . In general, the second-order rogue solution is composed of three first-order rogue waves. For the case where $c_1 = d_1 = 0$, the rogue wave are crowded round the origin (0,0) and the maximum of $u[2]$ and $v[2]$ is 5. For this case we have rogue wave composite (see figures 2 and 3), which resemble those obtained in Ref [20]. When we increase the value of $|c_1|$ and $|d_1|$ we observe that three first-order rogue waves are scattered in all direction (see figures 4-7).

Iterating formulae (25)-(28) three times (with $N = 3$), we obtain the third-order rogue wave solutions $u[3]$ and $v[3]$. We omit presenting analytical expressions since they are rather cumbersome to be write down here. By means of computer, we give the pictures of these third-order rogue wave solutions in figures 9 and 10. We know that the third-order rogue wave possesses four free parameters c_1 , d_1 , c_2 and d_2 . The third-order rogue wave solution is composed of six first-order rogue waves. For the case where $c_1 = d_1 = c_2 = d_2 = 0$, we can observe a composite third-order rogue wave solution and the maximum value of $u[3]$ and $v[3]$ is 7 (see figures 8 and 9).

When $c_1 = d_1 = 0$, $c_2 = d_2 = 100$, the six first-order rogue wave array a pentagon; among the six first-order rogue waves, one sits in the center and the rest are located on the vertices of the pentagon (see figures 10 and 11)

When $c_1 = d_1 = 100$, $c_2 = d_2 = 100$, the corresponding third-order rogue wave is composed of six first-order rogue waves as well, which array a triangle (see figures 12 and 13).

IV. CONCLUSION

In this paper, we presented in detail a procedure of the construction of a generalized DT for the Manakov system. This construction is divided into two steps. First, a brief introduction of the DT for the system (1) and (2) is given by the Darboux matrix method. Then, a detailed derivation of the generalized DT for system (1) and (2) is discussed through the Taylor expansion and a limit procedure. The generalized DT allows us to calculate the higher-order rogue wave solutions for the Manakov system (1) and (2) in a unified way. In particular, some higher-order rogue wave solutions for the Manakov system are constructed by means of the generalized DT with seed solutions $u[0] = e^{i\beta t}$ and $v[0] = e^{i\beta t}$.

The DT, which is originally the frame work of Darboux [14], is a powerful method for constructing solutions for integrable systems. In view to obtain rogue wave solutions for the nonlinear wave equations, the original DT is not applicable directly. So Matveev [15] introduced the so-called generalized DT and the positon solutions were calculated for the KdV equation [15]. Then, Guo et al. [13] re-examined Matveev's generalized DT and proposed a new approach

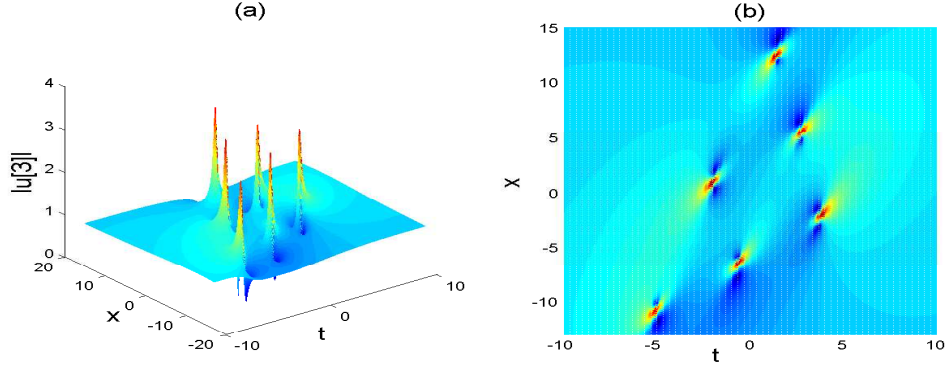


FIG. 12. Third-order rogue wave $u[3]$ depicted with the following parameters: $a = 1$, $b = 0$, $c_1 = d_1 = c_2 = d_2 = 100$ and $\alpha = 1$. Panel (a) represents the 3-D perspective and the panel (b) stands for the density plot of the 3-D representation

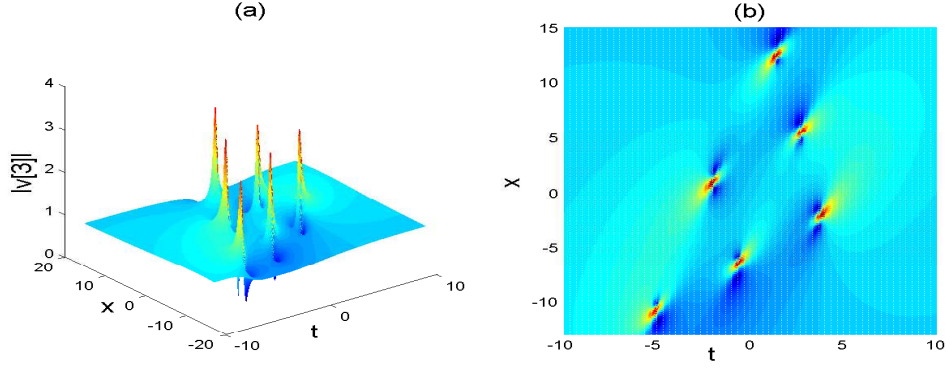


FIG. 13. Third-order rogue wave $v[3]$ depicted with the following parameters: $a = 0$, $b = 1$, $c_1 = d_1 = c_2 = d_2 = 100$ and $\alpha = 1$. Panel (a) represents the 3-D perspective and the panel (b) stands for the density plot of the 3-D representation

to derive the generalized DT for the KdV and the NLS equations. The Guo et al [13]'s approach has been used in this work to derive higher-order rogue wave for the Manakov system (1) and (2).

The first-order vector rogue solutions $u[1]$ and $v[1]$ are merely proportional and it is important to point out that their maximum is three times more than their each average crest. The profiles of the higher-order rogue wave solutions for the Manakov system depend on the values of their free parameters c_j and d_j ($j = 1, 2, 3, \dots$). The second-order vector rogue wave solutions $u[2]$ and $v[2]$ have their maximum five times more than their each average crest and are crowded round the origin $(0, 0)$ for the case where $c_1 = d_1 = 0$. For the case $c_1 = d_1 = 100$, we obtain three fundamental rogue waves scattered in all direction arraying a regular triangle and each fundamental rogue wave sits on a vertices. The third-order rogue waves $u[3]$ and $v[3]$ illustrated in this work have their maximum seven times more than each average crest and are crowded round the origin $(0, 0)$ for the case where $c_1 = d_1 = c_2 = d_2 = 0$. For the case $c_1 = d_1 = 0$ and $c_2 = d_2 = 100$, we obtain six fundamental rogue waves arraying a pentagon; among the six first-order rogue waves, one sits in the center and the rest are located on the vertices of the pentagon. If we set $c_1 = d_1 = c_2 = d_2 = 100$, we observe also six fundamental rogue waves scattered in all directions, arraying a regular triangle.

The first and the second-order rogue wave solutions illustrated in this paper resemble those obtained in Ref [20]. In this reference paper, the authors used a non-recursive DT. At our knowledge, the third-order rogue wave solutions for the Manakov system and a generalized formula for generating more than third-order have not yet been constructed. Thus, we actually believe that the present work is worth underlying and paves ways for further investigations of the system above within the viewpoint of the existence of propagation of extreme wave excitations. The observation of these rogue waves profiles is expected to be possible in Manakov or near-Manakov systems such as birefringent fiber transmission links ([22],[23]-[25]), crossing sea waves [17], photorefractive media [27], AlGaAs planar waveguide [26] and so on.

Moreover, the results obtained here can be helpful to describe the oceanic rogue waves in deep oceans and are expected to be useful for the technology of signal transmission through rogue wave. Further possible applications of our results can be found in the explanation of high intensity orthogonally polarized rogue light wave pulses in optical fibers. We hope our result will be realized by physical experiments in the future, which would be useful in the

understanding of the generation mechanisms.

V. REFERENCES

-
- [1] G. P. Agrawal, Nonlinear Fiber Optics (Academic Press, San Diego, 2006).
 - [2] A. Hasegawa and M. Matsumoto, Optical Solitons in Fibers (Springer-Verlag, Berlin, 2003).
 - [3] L. F. Mollenauer and J. P. Gordon, Solitons in Optical Fibers: Fundamentals and Applications (Academic Press, New York, 2006).
 - [4] K. Porsezian and V. C. Kuriakose, Optical Solitons: Theory and Experiment, Lecture Notes in Physics, Vol. 613 (Springer-Verlag, Berlin, 2003).
 - [5] Garrett C and Gemmrich J 2009 Phys. Today 7 62
 - [6] Solli D R, Ropers C, Koonath P and Jalali B 2007 Nature 450 1054
 - [7] Efimov V B, Ganshin A N, Kolmakov G V, McClintock P V E and Mezhev-Deglin L P 2010 Eur. Phys. J. Spec. Top. 185 181
 - [8] Bludov Yu V, Konotop V V and Akhmediev N 2009 Phys. Rev. A 80 033610
 - [9] Shats M, Punzmann H and Xia H 2010 Phys. Rev. Lett. 104 104503
 - [10] Peregrine D H 1983 J. Aust. Math. Soc. B 25 16
 - [11] Akhmediev N, Ankiewicz A and Soto-Crespo J M 2009 Phys. Rev. E 80 026601
 - [12] Akhmediev N, Ankiewicz A and Taki M 2009 Phys. Lett. A 373 675
 - [13] Guo B L, Ling L M and Liu Q P 2012 Phys. Rev. E 85 026607
 - [14] Darboux G 1882 C. R. Hebd. Seances Acad. Sci. 94 1456
 - [15] Matveev V B 1992 Phys. Lett. A 166 205
 - [16] Menyuk C R 1989 IEEE J. Quantum Electron. 25 2674
 - [17] Onorato M, Proment D and Toffoli A 2010 Eur. Phys. J. Spec. Top. 185 45
 - [18] Gu C H, Hu H S and Zhou Z X 2005 Darboux Transformations in Integrable Systems: Theory and their Applications to Geometry (Dordrecht: Springer)
 - [19] Zhaqilao 2013 Phys. Scr. 87 065401
 - [20] S. Chen and D. Mihalache 2015 J. Phys. A: Math. Theor. 48 215202
 - [21] V.B. Matveev, M.A. Salle 1991 Darboux Transformations and Solitons, Springer-Verlag, Berlin, Heidelberg
 - [22] Chen S, Soto-Crespo J M and Grelu P 2014 Opt. Express 22 27632
 - [23] Mihalache D, Mazilu D and Torner L 1998 Phys. Rev. Lett. 81 4353
 - [24] Torner L, Mihalache D, Mazilu D and Akhmediev N N 1997 Opt. Commun. 138 105
 - [25] Menyuk C R 1989 IEEE J. Quantum Electron. 25 2674
 - [26] Kang J U, Stegeman G I, Aitchison J S and Akhmediev N 1996 Phys. Rev. Lett. 76 3699
 - [27] Chen Z, Segev M, Coskun T, Christodoulides D N and Kivshar Y S 1997 J. Opt. Soc. Am. B 14 3066

VI. APPENDIX:

Analytical expressions of coefficients of equation (31)

$$\begin{aligned}
 r_1^{[0]} &= \frac{\sqrt{2}}{2\sqrt[4]{\beta}} (1-i) (-2\sqrt{\beta}x + 3\beta^{3/2}\alpha t - 4i\beta\alpha t - 1) \\
 &\times e^{-2\sqrt{\beta}x + 2\beta^{3/2}\alpha t - 4i\alpha\beta + 1/2i\beta t}, \\
 s_1^{[0]} &= \frac{\alpha\sqrt{2}}{2\beta^{3/4}} (1-i) (-2\sqrt{\beta}x + 3\beta^{3/2}\alpha t - 4i\beta\alpha t + 1) \\
 &\times e^{-2\sqrt{\beta}x + 2\beta^{3/2}\alpha t - 4i\alpha\beta - 1/2i\beta t}, \\
 w_1^{[0]} &= \frac{b\sqrt{2}}{2\beta^{3/4}} (1-i) (-2\sqrt{\beta}x + 3\beta^{3/2}\alpha t - 4i\beta\alpha t + 1) \\
 &\times e^{-2\sqrt{\beta}x + 2\beta^{3/2}\alpha t - 4i\alpha\beta - 1/2i\beta t}.
 \end{aligned}$$

For simplicity, we have put $\alpha = 1, (a = 1, b = 0 \text{ for } s_1^{[j]}, (a = 0, b = 1 \text{ for } w_1^{[j]}) (j=1,2)$

$$\begin{aligned}
 r_1^{[1]} &= (-1/24 + 1/24i) \sqrt{2} e^{-2x + 2t - 7/2it} (-3 + 117t^3 + 180xt + 15t - 42xt^2 + 147t^2 \\
 &- 18x - 36x^2 - 36x^2t + 24id_1 - 240ixt - 36it + 504it^2 - 144it^2x + 48ix^2t + 44it^3 + 24c_1 + 8x^3), \\
 s_1^{[1]} &= (1/24 - 1/24i) \sqrt{2} \\
 &\times e^{-2x + 2t - 9/2it} (-3 - 117t^3 - 252xt + 129t + 42xt^2 - 189t^2 - 30x + 60x^2 + 36x^2t - 24id_1 + 336ixt - 156it - 648it^2 + \\
 &144it^2x - 24c_1 - 8x^3 - 48ix^2t - 44it^3), \\
 w_1^{[1]} &= (1/24 - 1/24i) \sqrt{2} \\
 &\times e^{-2x + 2t - 9/2it} (-3 - 117t^3 - 252xt + 129t + 42xt^2 - 189t^2 - 30x + 60x^2 + 36x^2t - 24id_1 + 336ixt - 156it - 648it^2 +
 \end{aligned}$$

$$144it^2x - 24c_1 - 8x^3 - 48ix^2t - 44it^3).$$

$$r_1^{[2]} = \left(\frac{1}{960} - \frac{1}{960}i\right)\sqrt{2}$$

$$e^{-2x+2t-7/2it}(-45 - 210x - 765t + 11640x^2t - 3720xt + 21660xt^2 + 240x^4t + 60840t^3x - 4320tx^3 - 9240t^2x^2 + 5270t^4x + 560t^2x^3 - 4680t^3x^2 - 600x^2 + 3330t^2 - 1200x^3 - 102630t^3 - 39525t^4 + 560x^4 - 237t^5 - 32x^5 - 3840ixtc_1 + 960ixd_1 + 22880ixt^3 + 1680it^2d_1 + 1920it^2x^3 + 81120it^2x + 480ixt + 5760ix^3t + 5760itc_1 + 5760it^2c_1 + 2880xc_1t + 25200it^4 + 3116it^5 + 540it + 3840xd_1t - 320ix^4t - 31680it^2x^2 - 14880ix^2t - 4320itd_1 - 3360it^4x - 1760it^3x^2 - 960ix^2d_1 - 5760td_1 - 4320tc_1 - 5760t^2d_1 + 960xc_1 + 1680t^2c_1 - 960x^2c_1 - 41160it^3 - 19440it^2 - 240id_1 - 960id_2 + 2880ixtd_1 - 240c_1 - 960c_2),$$

$$s_1^{[2]} = \left(\frac{1}{960} - \frac{1}{960}i\right)\sqrt{2}$$

$$e^{-2x+2t-9/2it}(45 + 270x + 3555t + 23160x^2t - 19800xt + 38460xt^2 + 240x^4t + 70200t^3x - 5280tx^3 - 10920t^2x^2 + 5270t^4x + 560t^2x^3 - 4680t^3x^2 + 1560x^2 - 12450t^2 - 3120x^3 - 158790t^3 - 44795t^4 + 720x^4 - 237t^5 - 32x^5 - 30240ix^2t - 7200itd_1 - 37440it^2x^2 + 28560it^4 + 2880ixd_1 + 20640ixt + 9600itc_1 + 7040ix^3t + 26400ixt^3 + 138720it^2x - 3840ixtc_1 + 1680it^2d_1 + 1920it^2x^3 + 5760it^2c_1 + 2880xc_1t + 3116it^5 + 3840xd_1t - 320ix^4t - 3360it^4x - 1760it^3x^2 - 960ix^2d_1 - 62280it^3 - 1380it - 80400it^2 - 9600td_1 - 7200tc_1 - 5760t^2d_1 + 2880xc_1 + 1680t^2c_1 - 960x^2c_1 - 240id_1 - 960id_2 + 2880ixtd_1 - 240c_1 - 960c_2),$$

$$w_1^{[2]} = \left(\frac{1}{960} - \frac{1}{960}i\right)\sqrt{2}$$

$$e^{-2x+2t-9/2it}(45 + 270x + 3555t + 23160x^2t - 19800xt + 38460xt^2 + 240x^4t + 70200t^3x - 5280tx^3 - 10920t^2x^2 + 5270t^4x + 560t^2x^3 - 4680t^3x^2 + 1560x^2 - 12450t^2 - 3120x^3 - 158790t^3 - 44795t^4 + 720x^4 - 237t^5 - 32x^5 - 30240ix^2t - 7200itd_1 - 37440it^2x^2 + 28560it^4 + 2880ixd_1 + 20640ixt + 9600itc_1 + 7040ix^3t + 26400ixt^3 + 138720it^2x - 3840ixtc_1 + 1680it^2d_1 + 1920it^2x^3 + 5760it^2c_1 + 2880xc_1t + 3116it^5 + 3840xd_1t - 320ix^4t - 3360it^4x - 1760it^3x^2 - 960ix^2d_1 - 62280it^3 - 1380it - 80400it^2 - 9600td_1 - 7200tc_1 - 5760t^2d_1 + 2880xc_1 + 1680t^2c_1 - 960x^2c_1 - 240id_1 - 960id_2 + 2880ixtd_1 - 240c_1 - 960c_2).$$

Analytical expressions for $r_1[1]$, $s_1[1]$, $w_1[1]$, $r_1[2]$, $s_1[2]$ and $w_1[2]$.

$$r_1[1] = \frac{(1/6-1/6i)\sqrt{2}}{(-8x^2+24xt-50t^2-2)(-2x+3t+4it-1)}$$

$$e^{-2x+2t-7/2it}(-6 + 480itx^4 + 108t - 24x - 6100it^4x - 1952it^2x^3 - 168it^2c_1 - 576it^2d_1 - 480ix^2t + 384ixd_1t + 24d_1 - 288ixtc_1 + 96ix^2c_1 + 1776ixt^2 + 384xc_1t + 4464it^3x^2 - 1032it^3 - 64ix^5 - 12ix - 24ic_1 + 288xt d_1 - 3750t^4 - 48x^2 + 2900t^3 + 5000t^5 - 96x^4 - 300t^2 - 96x^3 + 3750it^5 + 64ix^3 + 66it + 128tx^4 + 168t^2d_1 - 4800t^4x - 768t^2x^3 - 1224xt^2 - 576t^2c_1 + 2752t^3x^2 + 240x^2t - 2064x^2t^2 + 3600xt^3 - 96x^2d_1 + 576x^3t + 144xt),$$

$$s_1[1] = \frac{(-1/6-1/6i)\sqrt{2}}{-8x^2+24xt-50t^2-2}e^{-2x+2t-9/2it}(12ix + 42t - 12x - 24c_1 - 350t^4 - 24x^2 - 366t^3 + 32x^4 - 486t^2 + 16x^3 + 204xt^2 - 168x^2t + 432x^2t^2 - 432xt^3 - 192x^3t + 168xt + 128ix^3t + 96itc_1 + 48ix^3 + 72tc_1 - 96td_1 - 1200it^4 - 24ix^2 - 198it^2 - 90it - 362it^3 - 24id_1 - 576it^2x^2 - 48ixd_1 - 120itx^2 - 24ixt + 72itd_1 + 228ixt^2 + 1376it^3x - 6i - 48xc_1),$$

$$w_1[1] = \frac{(-1/6-1/6i)\sqrt{2}}{-8x^2+24xt-50t^2-2}e^{-2x+2t-9/2it}(12ix + 42t - 12x - 24c_1 - 350t^4 - 24x^2 - 366t^3 + 32x^4 - 486t^2 + 16x^3 + 204xt^2 - 168x^2t + 432x^2t^2 - 432xt^3 - 192x^3t + 168xt + 128ix^3t + 96itc_1 + 48ix^3 + 72tc_1 - 96td_1 - 1200it^4 - 24ix^2 - 198it^2 - 90it - 362it^3 - 24id_1 - 576it^2x^2 - 48ixd_1 - 120itx^2 - 24ixt + 72itd_1 + 228ixt^2 + 1376it^3x - 6i - 48xc_1),$$

$$r_1[2] = (-1/30 - 1/30i)$$

$$\sqrt{2}e^{(-\frac{2}{65} + \frac{1}{130}i)(16x+28ix-65t)}(-540d_1 + 27648ix^5d_1t + 540x - 3510t - 4860x^2t + 14040xt + 1890xt^2 + 9408x^6t - 86016x^5t^3 - 12288x^7t + 44544x^6t^2 - 71136x^5t^2 + 36720t^2d_1^2 - 66960t^2c_1^2 + 8640x^2c_1^2 - 350280t^4x^3 + 436068t^5x^2 - 124800t^7x + 60672t^5x^3 - 140448t^6x^2 - 1313928t^5x - 1694730t^6x + 83520x^3t^3 - 10368x^5t - 36000x^4t^2 + 48960t^4x^4 + 35280x^4t - 437760t^3x - 17280tx^3 + 218160t^2x^2 - 2115270t^4x - 126000t^2x^3 + 628200t^3x^2 + 2880d_1x^4 - 8640d_1^2t + 8640x^2d_1^2 + 532440x^2t^4 - 234720t^4c_1 + 111852t^5d_1 - 129564t^5c_1 - 8064x^5c_1 + 432000c_1t^3 - 73440tc_1^2 + 8640xc_1^2 - 29160t^2d_1 + 244080t^2c_1 + 8640x^2c_1 - 16740td_1 + 15660tc_1 - 3240xc_1 + 29160t^3d_1 + 985140t^4d_1 + 276720x^4t^3 + 4320x^2d_1 - 145152x^6t^3 + 277440x^5t^4 - 6912x^8t + 41472x^7t^2 + 168480t^3d_1^2 - 121248t^5x^4 - 878176t^6x^3 + 11520x^3d_1^2 + 2523600t^7x^2 - 3288750t^8x - 123552t^6d_1 - 1078536t^6c_1 - 4608x^6c_1 + 1080xd_1 + 8640x^3d_2 + 2160xd_2 + 1080c_1 - 1080d_2 - 1080x^2 - 53190t^2 + 2520x^3 - 170325t^3 - 84480t^3x^3d_1 - 718560t^4x^2c_1 + 311040t^3x^3c_1 + 448704t^5xd_1 + 1315872t^5xc_1 - 9216x^5td_1 - 241920t^4x^2d_1 + 69120x^4t^2d_1 + 41472x^5tc_1 - 155520x^4t^2c_1 - 51840td_1^2x^2 - 60480t^2d_1^2x + 5760x^3td_1 - 25920xc_1t - 21600xd_1t + 108000x^2td_1 - 69120x^2tc_1 + 262440t^4xc_1 + 8640x^4td_1 + 48960x^4tc_1 + 538560c_1t^3x + 11520c_1tx^3 - 190080c_1t^2x^2 + 8640tc_1^2x - 60480xd_1^2t - 943200xt^3d_1 - 438480xt^2d_1 + 73440x^2t^2d_1 - 358920t^4xd_1 - 182880t^3x^2c_1 + 31680t^2x^3d_1 - 66240t^2x^3c_1 + 27360t^3x^2d_1 - 644130t^4 - 1440x^4 + 1790073t^5 - 4320x^5 + 2688x^6 - 26250t^8 + 8682t^6 + 2125893t^7 + 1536x^8 - 384x^7 - 2160d_1^2 - 2160c_1^2 + 1828125t^9 + 512x^9 - 35640t^2d_2 - 4320x^2c_2 + 2880x^3c_2 - 8640x^3d_1 - 5760x^5d_1 - 16200td_2 + 7560tc_2 - 2160xc_2 + 5760x^4c_2 - 87480t^2c_2 - 216000t^4d_2 - 65880t^3c_2 - 63000t^4c_2 - 65160t^3d_2 - 69120x^2tc_1d_1 + 207360t^2c_1xd_1 + 51840xtc_1d_1 - 63360t^3d_1c_1 - 17280x^2c_1d_1 + 8640xc_1d_1 + 12960tc_1c_2 + 247680t^3xd_2 + 30240t^2c_1d_1 - 2160ixc_2 + 23040x^3td_2 + 36720xt^2c_2 - 4320xt d_2 - 21600x^2td_2 - 34560x^3tc_2 + 30240xtc_2 - 30240x^2tc_2 - 103680t^2x^2d_2 + 77760t^2x^2c_2 - 77760t^3xc_2 + 12960td_1d_2 + 38880c_1td_1 + 41040t^2xd_2 + 760320ix^3t^3d_1 + 56160ixtc_1 + 77760it^2x^2d_2 + 8640ixd_1c_2 + 4320ixtc_2 + 60480ixtd_1^2 + 21600ix^2tc_2 + 64800ixt^2d_1 + 8640ixc_1^2t + 4320id_1^2x + 4320ix^2c_2 + 8640id_1c_1^2 + 244800ic_1t^2x^3 + 79920it^3d_1 + 4320id_1c_2 + 228960it^4d_1 + 23760itc_1^2 + 62640id_1^2t + 35640it^2c_2 + 5760ix^4d_2 + 216000it^4c_2 + 63360it^3c_1^2 + 286200it^2d_1 + 4320ix^2d_1 + 226304ix^6t^3 + 1582920it^4x^3 + 32940ix^2t + 315000it^3x^2 + 2292864it^5x^4 +$$

$$\begin{aligned}
& 112608 ix^5 t^2 + 3072 ix^8 t + 4795200 it^7 x^2 + 1102890 it^6 x + 19440 ix^4 t + 65160 it^3 c_2 + 2036400 it^7 x + 89280 it^3 x + 15360 ix^7 t + \\
& 832104 it^5 x + 1802304 it^5 x^3 + 563400 ix^2 t^4 + 2880 ix^3 d_2 + 6480 it^2 x^2 + 16200 itc_2 + 7560 itd_2 + 8640 ic_1 x^3 + 18720 ix^4 t^2 + \\
& 5760 ix^5 t + 398592 ix^5 t^3 + 1080 ixt - 23040 ix^3 tc_2 - 315360 ic_1 t^2 x^2 - 195840 ix^4 t^2 d_1 - 168480 it^3 d_1 c_1 - 207360 it^2 c_1^2 x - \\
& 8640 ic_1 x d_2 - 241920 ix^2 t^4 c_1 - 9216 ix^5 c_1 t - 34560 itx^3 d_2 - 103680 ic_1 t^2 d_1 - 8640 ix^2 t d_1 - 41040 it^2 x c_2 - 54720 itx^4 c_1 - \\
& 124920 it^4 c_1 x - 38880 itc_1 x^2 - 496440 ixt^4 d_1 - 280800 ic_1 t^3 x^2 - 146880 it^2 x^3 d_1 - 30240 ix^2 t d_2 + 8640 ix^2 c_1^2 + \\
& 5760 ix^5 c_1 + 149400 ic_1 t^3 + 336852 it^5 c_1 + 8100 itc_1 - 247680 it^3 x c_2 - 11520 ic_1 x^3 d_1 - 64800 itc_1 d_1 - 1477440 ix^2 t^4 d_1 - \\
& 84480 ix^3 t^3 c_1 - 17280 itd_1 d_2 - 17280 itc_1 c_2 - 155520 ix^2 t^2 d_1 - 12960 ix d_1 t - 12960 itd_1 c_2 - 77760 it^3 x d_2 - \\
& 65880 it^3 d_2 - 66960 it^2 c_1^2 - 8640 ix^3 c_2 - 95436 it^5 d_1 - 4320 ix c_1^2 - 128250 ixt^2 - 1392210 it^4 x - 36864 ix^7 t^2 - \\
& 875520 ix^5 t^4 - 4108032 it^6 x^3 - 585840 ix^4 t^3 - 14016 ix^6 t - 1901124 it^5 x^2 - 43920 it^2 x^3 - 3240000 it^8 x - 3072 ix^6 d_1 - \\
& 4320 ix^2 c_1 - 4320 ix^2 d_2 - 63000 it^4 d_2 - 160920 it^2 c_1 - 2880 ix^4 c_1 - 774900 ic_1 t^4 - 87480 it^2 d_2 - 796464 it^6 d_1 - \\
& 123552 it^6 c_1 - 36720 it^2 d_1^2 - 4320 ic_1 d_2 - 3456 ix^5 d_1 - 2160 ix d_2 - 8640 ix^2 d_1^2 - 1080 ic_1 x - 1620 itd_1 - 5760 ix^3 d_1 - \\
& 2251536 it^6 x^2 - 17280 ix^3 t - 100608 ix^6 t^2 - 1013280 ix^4 t^4 - 348480 ix^3 t^3 - 3240 ix d_1 + 14400 itx^4 d_1 + 17280 ic_1 x^3 t + \\
& 12960 ic_1 t d_2 + 448704 it^5 x c_1 + 69120 ix^2 c_1^2 t + 1080 ic_2 + 8640 id_1^3 + 69120 ix^4 t^2 c_1 + 384 ix^7 + 540 ic_1 + 1080 id_1 + \\
& 30240 ixt d_2 + 69120 itc_1 x d_1 + 51840 ix^2 d_1 t c_1 - 135 i + 60480 it^2 d_1 x c_1 + 92880 ic_1 t^2 x + 8640 ix c_1 d_1 - 4320 x^2 d_2 - \\
& 4320 c_1 c_2 - 4320 d_1 d_2 - 768 ix^8 - 3960 ix^3 - 270 ix - 3744 ix^5 - 1094349 it^7 - 1261875 it^8 - 327870 it^4 - 2128626 it^6 - \\
& 1152 ix^6 - 33750 it^2 - 1080 ix^2 + 8640 c_1 d_1^2 + 2880 ixt^3 d_1 + 36720 ixt^2 d_2 + 687500 it^9 + 457065 it^3 + 1135251 it^5 + \\
& 1485 it + 1440 ix^4 - 8640 x d_1 d_2 - 8640 x c_1 c_2 - 17280 t d_1 c_2 + 17280 c_1 t d_2 + 1236960 ic_1 t^3 x + 103680 it^2 x^2 c_2 + \\
& 8640 c_1^3 + 491040 it^3 x^2 d_1 + 1384128 ixt^5 d_1 + 34560 itd_1 x^3) / (9 - 612 xt - 192 x^3 c_1 - 22500 t^5 x - 8928 x^3 t^3 - 576 x^5 t + \\
& 2928 x^4 t^2 + 792 t^3 x + 96 tx^3 - 2232 t^2 x^2 + 18300 x^2 t^4 - 2808 c_1 t^3 + 864 t d_1 - 792 t c_1 + 144 x c_1 - 1056 t^3 d_1 + 108 x^2 + \\
& 2835 t^2 - 1152 x^2 t d_1 + 864 x^2 t c_1 + 1008 c_1 x t^2 + 3456 x t^2 d_1 + 10179 t^4 + 48 x^4 + 64 x^6 + 15625 t^6 + 144 d_1^2 + 144 c_1^2),
\end{aligned}$$

$$\begin{aligned}
& s_1[2] = (-1/30 - 1/30 i) \\
& \sqrt{2}e^{(-\frac{2}{97} + \frac{9}{194} i)(16x + 36ix - 97t)} (-540 d_1 + 95436 it^5 d_1 + 8640 ix^3 c_2 + 66960 it^2 c_1^2 + 65880 it^3 d_2 + 3240 ix d_1 + 348480 ix^3 t^3 + \\
& 1013280 ix^4 t^4 + 100608 ix^6 t^2 + 17280 ix^3 t + 2251536 it^6 x^2 + 5760 ix^3 d_1 + 1620 itd_1 + 1080 ic_1 x + 2160 ix d_2 + \\
& 3456 ix^5 d_1 + 4320 ic_1 d_2 + 36720 it^2 d_1^2 - 491040 it^3 x^2 d_1 - 244800 ic_1 t^2 x^3 - 92880 ic_1 t^2 x - 8640 ix c_1^2 t + 27648 ix^5 d_1 t + \\
& 103680 ic_1 t^2 d_1 + 41040 it^2 x c_2 + 540 x - 3510 t + 8640 ix^2 t d_1 + 496440 ixt^4 d_1 + 38880 itc_1 x^2 + 124920 it^4 c_1 x + \\
& 54720 itx^4 c_1 + 30240 ix^2 t d_2 + 146880 it^2 x^3 d_1 + 280800 ic_1 t^3 x^2 - 4860 x^2 t - 14040 xt + 1890 xt^2 + 9408 x^6 t + 86016 x^5 t^3 + \\
& 12288 x^7 t - 44544 x^6 t^2 - 71136 x^5 t^2 - 36720 t^2 d_1^2 + 66960 t^2 c_1^2 - 8640 x^2 c_1^2 - 350280 t^4 x^3 + 436068 t^5 x^2 + 124800 t^7 x - \\
& 60672 t^5 x^3 + 140448 t^6 x^2 + 1313928 t^5 x - 1694730 t^6 x - 83520 x^3 t^3 + 10368 x^5 t + 36000 x^4 t^2 - 48960 t^4 x^4 + 35280 x^4 t + \\
& 437760 t^3 x + 17280 tx^3 - 218160 t^2 x^2 - 2115270 t^4 x - 126000 t^2 x^3 + 628200 t^3 x^2 + 2880 d_1 x^4 - 8640 d_1^2 t - 8640 x^2 d_1^2 - \\
& 532440 x^2 t^4 - 234720 t^4 c_1 - 111852 t^5 d_1 + 129564 t^5 c_1 + 8064 x^5 c_1 - 432000 c_1 t^3 - 73440 tc_1^2 + 8640 xc_1^2 - 29160 t^2 d_1 + \\
& 244080 t^2 c_1 + 8640 x^2 c_1 + 16740 t d_1 - 15660 tc_1 + 3240 xc_1 - 29160 t^3 d_1 + 985140 t^4 d_1 + 276720 x^4 t^3 + 4320 x^2 d_1 - \\
& 145152 x^6 t^3 + 277440 x^5 t^4 - 6912 x^8 t + 41472 x^7 t^2 + 168480 t^3 d_1^2 - 121248 t^5 x^4 - 878176 t^6 x^3 + 11520 x^3 d_1^2 + \\
& 2523600 t^7 x^2 - 3288750 t^8 x - 123552 t^6 d_1 - 1078536 t^6 c_1 - 4608 x^6 c_1 - 1080 x d_1 - 8640 x^3 d_2 - 2160 x d_2 + 1080 c_1 - \\
& 1080 d_2 + 1080 x^2 + 53190 t^2 + 2520 x^3 - 170325 t^3 - 84480 t^3 x^3 d_1 - 718560 t^4 x^2 c_1 + 311040 t^3 x^3 c_1 + 448704 t^5 x d_1 + \\
& 1315872 t^5 x c_1 - 9216 x^5 t d_1 - 241920 t^4 x^2 d_1 + 69120 x^4 t^2 d_1 + 41472 x^5 t c_1 - 155520 x^4 t^2 c_1 - 51840 t d_1^2 x^2 - 60480 t^2 d_1^2 x + \\
& 5760 x^3 t d_1 - 25920 xc_1 t - 21600 x d_1 t - 108000 x^2 t d_1 + 69120 x^2 t c_1 - 262440 t^4 x c_1 - 8640 x^4 t d_1 - 48960 x^4 t c_1 + \\
& 538560 c_1 t^3 x + 11520 c_1 t x^3 - 190080 c_1 t^2 x^2 - 8640 tc_1^2 x + 60480 x d_1^2 t - 943200 xt^3 d_1 + 438480 xt^2 d_1 + 73440 x^2 t^2 d_1 + \\
& 358920 t^4 x d_1 + 182880 t^3 x^2 c_1 - 31680 t^2 x^3 d_1 + 66240 t^2 x^3 c_1 - 27360 t^3 x^2 d_1 + 644130 t^4 + 1440 x^4 + 1790073 t^5 - \\
& 4320 x^5 - 2688 x^6 + 26250 t^8 - 8682 t^6 + 2125893 t^7 - 1536 x^8 - 384 x^7 + 2160 d_1^2 + 2160 c_1^2 + 1828125 t^9 + 512 x^9 - \\
& 35640 t^2 d_2 - 4320 x^2 c_2 - 2880 x^3 c_2 + 8640 x^3 d_1 + 5760 x^5 d_1 + 16200 t d_2 - 7560 tc_2 + 2160 xc_2 + 5760 x^4 c_2 - 87480 t^2 c_2 - \\
& 216000 t^4 d_2 + 65880 t^3 c_2 - 63000 t^4 c_2 + 65160 t^3 d_2 - 69120 x^2 t c_1 d_1 + 207360 t^2 c_1 x d_1 - 60480 ixt d_1^2 - 51840 xtc_1 d_1 - \\
& 63360 t^3 d_1 c_1 + 17280 x^2 c_1 d_1 + 8640 xc_1 d_1 + 12960 tc_1 c_2 + 247680 t^3 x d_2 - 30240 t^2 c_1 d_1 + 8640 it^2 d_1^2 - 8640 it^2 c_1^2 + \\
& 23040 x^3 t d_2 - 36720 x^2 t c_2 - 4320 x t d_2 + 21600 x^2 t d_2 - 34560 x^3 t c_2 + 30240 x t c_2 + 30240 x^2 t c_2 - 103680 t^2 x d_2 + \\
& 77760 t^2 x^2 c_2 - 77760 t^3 x c_2 + 12960 t d_1 d_2 + 38880 c_1 t d_1 - 41040 t^2 x d_2 + 760320 ix^3 t^3 d_1 - 21600 ix^2 t c_2 - 14400 itx^4 d_1 - \\
& 64800 ixt^2 d_1 - 36720 ixt^2 d_2 + 56160 ixt c_1 + 77760 it^2 x^2 d_2 + 8640 ix d_1 c_2 + 4320 ixt c_2 - 1440 ix^4 + 1080 ix^2 + 33750 it^2 + \\
& 1152 ix^6 + 2128626 it^6 + 327870 it^4 + 1261875 it^8 + 768 ix^8 - 69120 itc_1 x d_1 + 4320 id_1^2 x + 4320 ix^2 c_2 + 8640 id_1 c_1^2 + \\
& 228960 it^4 d_1 + 23760 itc_1^2 + 62640 id_1^2 t + 35640 it^2 c_2 + 5760 ix^4 d_2 + 216000 it^4 c_2 + 63360 it^3 c_1^2 + 286200 it^2 d_1 + \\
& 4320 ix^2 d_1 + 226304 ix^6 t^3 + 1582920 it^4 x^3 + 32940 ix^2 t + 315000 it^3 x^2 + 2292864 it^5 x^4 + 112608 ix^5 t^2 + 3072 ix^8 t + \\
& 4795200 it^7 x^2 + 1102890 it^6 x + 19440 ix^4 t - 23040 ix^3 t c_2 - 315360 ic_1 t^2 x^2 - 195840 ix^4 t^2 d_1 - 168480 it^3 d_1 c_1 - \\
& 207360 it^2 c_1^2 x - 8640 ic_1 x d_2 - 241920 ix^2 t^4 c_1 - 9216 ix^5 c_1 t - 34560 itx^3 d_2 - 247680 it^3 x c_2 - 11520 ic_1 x^3 d_1 - \\
& 64800 itc_1 d_1 - 1477440 ix^2 t^4 d_1 - 84480 ix^3 t^3 c_1 - 17280 itd_1 d_2 - 17280 itc_1 c_2 - 155520 ix^2 t^2 d_1 - 12960 ix d_1 t - \\
& 12960 itd_1 c_2 - 77760 it^3 x d_2 - 4320 ix c_1^2 - 128250 ixt^2 - 1392210 it^4 x - 36864 ix^7 t^2 - 875520 ix^5 t^4 - 4108032 it^6 x^3 - \\
& 585840 ix^4 t^3 - 14016 ix^6 t - 1901124 it^5 x^2 - 43920 it^2 x^3 - 3240000 it^8 x - 3072 ix^6 d_1 - 4320 ix^2 c_1 - 4320 ix^2 d_2 - \\
& 63000 it^4 d_2 - 160920 it^2 c_1 - 2880 ix^4 c_1 - 774900 ic_1 t^4 - 87480 it^2 d_2 - 796464 it^6 d_1 - 123552 it^6 c_1 + 17280 ic_1 x^3 t + \\
& 12960 ic_1 t d_2 + 448704 it^5 x c_1 + 69120 ix^2 c_1^2 t + 1080 ic_2 + 8640 id_1^3 + 69120 ix^4 t^2 c_1 + 384 ix^7 + 540 ic_1 + 1080 id_1 + \\
& 30240 ixt d_2 + 51840 ix^2 d_1 t c_1 + 60480 it^2 d_1 x c_1 + 8640 ix c_1 d_1 - 4320 x^2 d_2 + 4320 c_1 c_2 + 4320 d_1 d_2 - 3960 ix^3 - 270 ix - \\
& 3744 ix^5 - 1094349 it^7 + 8640 c_1 d_1^2 + 2160 ix c_2 + 2880 ixt^3 d_1 + 687500 it^9 + 457065 it^3 + 1135251 it^5 + 1485 it + 135 i - \\
& 8640 x d_1 d_2 - 8640 x c_1 c_2 - 17280 t d_1 c_2 + 17280 c_1 t d_2 + 1236960 ic_1 t^3 x + 103680 it^2 x^2 c_2 - 4320 id_1 c_2 - 6480 it^2 x^2 - \\
& 563400 ix^2 t^4 - 1802304 it^5 x^3 - 832104 it^5 x - 15360 ix^7 t - 89280 it^3 x - 2036400 it^7 x - 1080 ixt - 398592 ix^5 t^3 - 5760 ix^5 t -
\end{aligned}$$

$$\begin{aligned}
& 18720 ix^4 t^2 - 336852 it^5 c_1 - 8100 itc_1 - 65160 it^3 c_2 - 2880 ix^3 d_2 - 7560 itd_2 - 16200 itc_2 - 5760 ix^5 c_1 - 8640 ic_1 x^3 - \\
& 149400 ic_1 t^3 - 79920 it^3 d_1 + 8640 c_1^3 + 1384128 itx^5 d_1 + 34560 itd_1 x^3)/(9 - 612 xt - 192 x^3 c_1 - 22500 t^5 x - 8928 x^3 t^3 - \\
& 576 x^5 t + 2928 x^4 t^2 + 792 t^3 x + 96 tx^3 - 2232 t^2 x^2 + 18300 x^2 t^4 - 2808 c_1 t^3 + 864 td_1 - 792 tc_1 + 144 xc_1 - 1056 t^3 d_1 + \\
& 108 x^2 + 2835 t^2 - 1152 x^2 td_1 + 864 x^2 tc_1 + 1008 c_1 xt^2 + 3456 xt^2 d_1 + 10179 t^4 + 48 x^4 + 64 x^6 + 15625 t^6 + 144 d_1^2 + 144 c_1^2), \\
& w_1[2] = (-1/30 - 1/30 i) \\
& \sqrt{2}e^{(-\frac{9}{57} + \frac{9}{154} i)(16x + 36ix - 97t)} (-540 d_1 + 95436 it^5 d_1 + 8640 ix^3 c_2 + 66960 it^2 c_1^2 + 65880 it^3 d_2 + 3240 ix d_1 + 348480 ix^3 t^3 + \\
& 1013280 ix^4 t^4 + 100608 ix^6 t^2 + 17280 ix^3 t + 2251536 it^6 x^2 + 5760 ix^3 d_1 + 1620 itd_1 + 1080 ic_1 x + 2160 ix d_2 + \\
& 3456 ix^5 d_1 + 4320 ic_1 d_2 + 36720 it^2 d_1^2 - 491040 it^3 x^2 d_1 - 244800 ic_1 t^2 x^3 - 92880 ic_1 t^2 x - 8640 ix c_1^2 t + 27648 ix^5 d_1 t + \\
& 103680 ic_1 t^2 d_1 + 41040 it^2 xc_2 + 540 x - 3510 t + 8640 ix^2 td_1 + 496440 itx^4 d_1 + 38880 itc_1 x^2 + 124920 it^4 c_1 x + \\
& 54720 itx^4 c_1 + 30240 ix^2 td_2 + 146880 it^2 x^3 d_1 + 280800 ic_1 t^3 x^2 - 4860 x^2 t - 14040 xt + 1890 xt^2 + 9408 x^6 t + 86016 x^5 t^3 + \\
& 12288 x^7 t - 44544 x^6 t^2 - 71136 x^5 t^2 - 36720 t^2 d_1^2 + 66960 t^2 c_1^2 - 8640 x^2 c_1^2 - 350280 t^4 x^3 + 436068 t^5 x^2 + 124800 t^7 x - \\
& 60672 t^5 x^3 + 140448 t^6 x^2 + 1313928 t^5 x - 1694730 t^6 x - 83520 x^3 t^3 + 10368 x^5 t + 36000 x^4 t^2 - 48960 t^4 x^4 + 35280 x^4 t + \\
& 437760 t^3 x + 17280 tx^3 - 218160 t^2 x^2 - 2115270 t^4 x - 126000 t^2 x^3 + 628200 t^3 x^2 + 2880 d_1 x^4 - 8640 d_1^2 t - 8640 x^2 d_1^2 - \\
& 532440 x^2 t^4 - 234720 t^4 c_1 - 111852 t^5 d_1 + 129564 t^5 c_1 + 8064 x^5 c_1 - 432000 c_1 t^3 - 73440 tc_1^2 + 8640 xc_1^2 - 29160 t^2 d_1 + \\
& 244080 t^2 c_1 + 8640 x^2 c_1 + 16740 td_1 - 15660 tc_1 + 3240 xc_1 - 29160 t^3 d_1 + 985140 t^4 d_1 + 276720 x^4 t^3 + 4320 x^2 d_1 - \\
& 145152 x^6 t^3 + 277440 x^5 t^4 - 6912 x^8 t + 41472 x^7 t^2 + 168480 t^3 d_1^2 - 121248 t^5 x^4 - 878176 t^6 x^3 + 11520 x^3 d_1^2 + \\
& 2523600 t^7 x^2 - 3288750 t^8 x - 123552 t^6 d_1 - 1078536 t^6 c_1 - 4608 x^6 c_1 - 1080 xd_1 - 8640 x^3 d_2 - 2160 xd_2 + 1080 c_1 - \\
& 1080 d_2 + 1080 x^2 + 53190 t^2 + 2520 x^3 - 170325 t^3 - 84480 t^3 x^3 d_1 - 718560 t^4 x^2 c_1 + 311040 t^3 x^3 c_1 + 448704 t^5 x d_1 + \\
& 1315872 t^5 xc_1 - 9216 x^5 td_1 - 241920 t^4 x^2 d_1 + 69120 x^4 t^2 d_1 + 41472 x^5 tc_1 - 155520 x^4 t^2 c_1 - 51840 td_1^2 x^2 - 60480 t^2 d_1^2 x + \\
& 5760 x^3 td_1 - 25920 xc_1 t - 21600 xd_1 t - 108000 x^2 td_1 + 69120 x^2 tc_1 - 262440 t^4 xc_1 - 8640 x^4 td_1 - 48960 x^4 tc_1 + \\
& 538560 c_1 t^3 x + 11520 c_1 tx^3 - 190080 c_1 t^2 x^2 - 8640 tc_1^2 x + 60480 x d_1^2 t - 943200 xt^3 d_1 + 438480 xt^2 d_1 + 73440 x^2 t^2 d_1 + \\
& 358920 t^4 x d_1 + 182880 t^3 x^2 c_1 - 31680 t^2 x^3 d_1 + 66240 t^2 x^3 c_1 - 27360 t^3 x^2 d_1 + 644130 t^4 + 1440 x^4 + 1790073 t^5 - \\
& 4320 x^5 - 2688 x^6 + 26250 t^8 - 8682 t^6 + 2125893 t^7 - 1536 x^8 - 384 x^7 + 2160 d_1^2 + 2160 c_1^2 + 1828125 t^9 + 512 x^9 - \\
& 35640 t^2 d_2 - 4320 x^2 c_2 - 2880 x^3 c_2 + 8640 x^3 d_1 + 5760 x^5 d_1 + 16200 td_2 - 7560 tc_2 + 2160 xc_2 + 5760 x^4 c_2 - 87480 t^2 c_2 - \\
& 216000 t^4 d_2 + 65880 t^3 c_2 - 63000 t^4 c_2 + 65160 t^3 d_2 - 69120 x^2 tc_1 d_1 + 207360 t^2 c_1 x d_1 - 60480 itx d_1^2 - 51840 xtc_1 d_1 - \\
& 63360 t^3 d_1 c_1 + 17280 x^2 c_1 d_1 + 8640 xc_1 d_1 + 12960 tc_1 c_2 + 247680 t^3 x d_2 - 30240 t^2 c_1 d_1 + 8640 ix^2 d_1^2 - 8640 ix^2 c_1^2 + \\
& 23040 x^3 td_2 - 36720 xt^2 c_2 - 4320 xtd_2 + 21600 x^2 td_2 - 34560 x^3 tc_2 + 30240 xtc_2 + 30240 x^2 tc_2 - 103680 t^2 x^2 d_2 + \\
& 77760 t^2 x^2 c_2 - 77760 t^3 xc_2 + 12960 td_1 d_2 + 38880 c_1 td_1 - 41040 t^2 x d_2 + 760320 ix^3 t^3 d_1 - 21600 ix^2 tc_2 - 14400 itx^4 d_1 - \\
& 64800 ix^2 t^2 d_1 - 36720 itx^2 d_2 + 56160 itxc_1 + 77760 it^2 x^2 d_2 + 8640 ix d_1 c_2 + 4320 itxc_2 - 1440 ix^4 + 1080 ix^2 + 33750 it^2 + \\
& 1152 ix^6 + 2128626 it^6 + 327870 it^4 + 1261875 it^8 + 768 ix^8 - 69120 itc_1 x d_1 + 4320 id_1^2 x + 4320 ix^2 c_2 + 8640 id_1 c_1^2 + \\
& 228960 it^4 d_1 + 23760 itc_1^2 + 62640 id_1^2 t + 35640 it^2 c_2 + 5760 ix^4 d_2 + 216000 it^4 c_2 + 63360 it^3 c_1^2 + 286200 it^2 d_1 + \\
& 4320 ix^2 d_1 + 226304 ix^6 t^3 + 1582920 it^4 x^3 + 32940 ix^2 t + 315000 it^3 x^2 + 2292864 it^5 x^4 + 112608 ix^5 t^2 + 3072 ix^8 t + \\
& 4795200 it^7 x^2 + 1102890 it^6 x + 19440 ix^4 t - 23040 ix^3 tc_2 - 315360 ic_1 t^2 x^2 - 195840 ix^4 t^2 d_1 - 168480 it^3 d_1 c_1 - \\
& 207360 it^2 c_1^2 x - 8640 ic_1 x d_2 - 241920 ix^2 t^4 c_1 - 9216 ix^5 c_1 t - 34560 itx^3 d_2 - 247680 it^3 xc_2 - 11520 ic_1 x^3 d_1 - \\
& 64800 itc_1 d_1 - 1477440 ix^2 t^4 d_1 - 84480 ix^3 t^3 c_1 - 17280 itd_1 d_2 - 17280 itc_1 c_2 - 155520 ix^2 t^2 d_1 - 12960 ix d_1 t - \\
& 12960 it d_1 c_2 - 77760 it^3 x d_2 - 4320 ix c_1^2 - 128250 itx^2 - 1392210 it^4 x - 36864 ix^7 t^2 - 875520 ix^5 t^4 - 4108032 it^6 x^3 - \\
& 585840 ix^4 t^3 - 14016 ix^6 t - 1901124 it^5 x^2 - 43920 it^2 x^3 - 3240000 it^8 x - 3072 ix^6 d_1 - 4320 ix^2 c_1 - 4320 ix^2 d_2 - \\
& 63000 it^4 d_2 - 160920 it^2 c_1 - 2880 ix^4 c_1 - 774900 ic_1 t^4 - 87480 it^2 d_2 - 796464 it^6 d_1 - 123552 it^6 c_1 + 17280 ic_1 x^3 t + \\
& 12960 ic_1 td_2 + 448704 it^5 xc_1 + 69120 ix^2 c_1^2 t + 1080 ic_2 + 8640 id_1^3 + 69120 ix^4 t^2 c_1 + 384 ix^7 + 540 ic_1 + 1080 id_1 + \\
& 30240 itx d_2 + 51840 ix^2 d_1 tc_1 + 60480 it^2 d_1 xc_1 + 8640 ix c_1 d_1 - 4320 x^2 d_2 + 4320 c_1 c_2 + 4320 d_1 d_2 - 3960 ix^3 - 270 ix - \\
& 3744 ix^5 - 1094349 it^7 + 8640 c_1 d_1^2 + 2160 ix c_2 + 2880 itx^3 d_1 + 687500 it^9 + 457065 it^3 + 1135251 it^5 + 1485 it + 135 i - \\
& 8640 xd_1 d_2 - 8640 xc_1 c_2 - 17280 td_1 c_2 + 17280 c_1 td_2 + 1236960 ic_1 t^3 x + 103680 it^2 x^2 c_2 - 4320 id_1 c_2 - 6480 it^2 x^2 - \\
& 563400 ix^2 t^4 - 1802304 it^5 x^3 - 832104 it^5 x - 15360 ix^7 t - 89280 it^3 x - 2036400 it^7 x - 1080 itx - 398592 ix^5 t^3 - 5760 ix^5 t - \\
& 18720 ix^4 t^2 - 336852 it^5 c_1 - 8100 itc_1 - 65160 it^3 c_2 - 2880 ix^3 d_2 - 7560 itd_2 - 16200 itc_2 - 5760 ix^5 c_1 - 8640 ic_1 x^3 - \\
& 149400 ic_1 t^3 - 79920 it^3 d_1 + 8640 c_1^3 + 1384128 itx^5 d_1 + 34560 itd_1 x^3)/(9 - 612 xt - 192 x^3 c_1 - 22500 t^5 x - 8928 x^3 t^3 - \\
& 576 x^5 t + 2928 x^4 t^2 + 792 t^3 x + 96 tx^3 - 2232 t^2 x^2 + 18300 x^2 t^4 - 2808 c_1 t^3 + 864 td_1 - 792 tc_1 + 144 xc_1 - 1056 t^3 d_1 + \\
& 108 x^2 + 2835 t^2 - 1152 x^2 td_1 + 864 x^2 tc_1 + 1008 c_1 xt^2 + 3456 xt^2 d_1 + 10179 t^4 + 48 x^4 + 64 x^6 + 15625 t^6 + 144 d_1^2 + 144 c_1^2).
\end{aligned}$$